Year 12 Mathematics: Specialist
Term 2 2020

SHENTON
COLLEGE

Test 2 *Calculator Free*
Functions, Graphs & Vectors in 3D

Student Name: Solutions (FINAL)

Teacher: Alfonsi Moore

Working Time: 30 minutes Formula Sheet provided.

*Attempt **all** questions.*
*All necessary working and reasoning must be shown for **full marks**.*

Total Marks
$\frac{37}{37}$

Question 1. **(4 marks)**

Solve the following system of linear equations using Gaussian elimination.

$$x - y + 2z = 4$$
$$-2x - y + 3z = -1$$
$$4x - y - z = 7$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ -2 & -1 & 3 & -1 \\ 4 & -1 & -1 & 7 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \Rightarrow R_2 + 2R_1 \\ R_3 \Rightarrow R_3 - 4R_1 \end{array}$$

✓ obtains new R_2 and R_3 using elementary row operations

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 0 & -3 & 7 & 7 \\ 0 & 3 & -9 & -9 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 \Rightarrow R_3 + R_2 \end{array}$$

✓ reduces to echelon form using elementary row operations.

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 0 & -3 & 7 & 7 \\ 0 & 0 & -2 & -2 \end{array}\right]$$

$\therefore -2z = -2$ $\Rightarrow -3y + 7(1) = 7$ $\Rightarrow x - 0 + 2(1) = 4$

$\underline{\underline{z = 1}}$ $3y = 0$ $x + 2 = 4$

$\underline{\underline{y = 0}}$ $\underline{\underline{x = 2}}$

$\therefore (x, y, z) = (2, 0, 1)$ //

✓✓ correct solution for x, y and z.

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Question 2. **(8 marks)**

Two functions, f and g, are defined as:

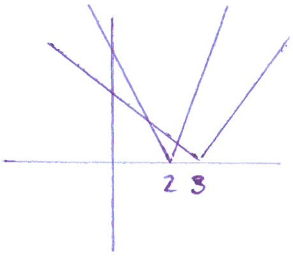
$$f(x) = |x| + |2x + 2| \qquad g(x) = x - 3$$

(a) State $f(g(x))$, simplifying where possible. (1 mark)

$f(g(x)) = |x-3| + |2(x-3)+2|$

$= |x-3| + |2x-4|$ // ✓ determines $f(g(x))$ correctly.

(b) Hence, determine the piecewise definition of $f(g(x))$. (3 marks)



For $x \leq 2$, $f(g(x)) = -x+3-2x+4$

$= -3x+7$

For $2 < x \leq 3$, $f(g(x)) = -x+3+2x-4$

$= x-1$

For $x > 3$, $f(g(x)) = x-3+2x-4$

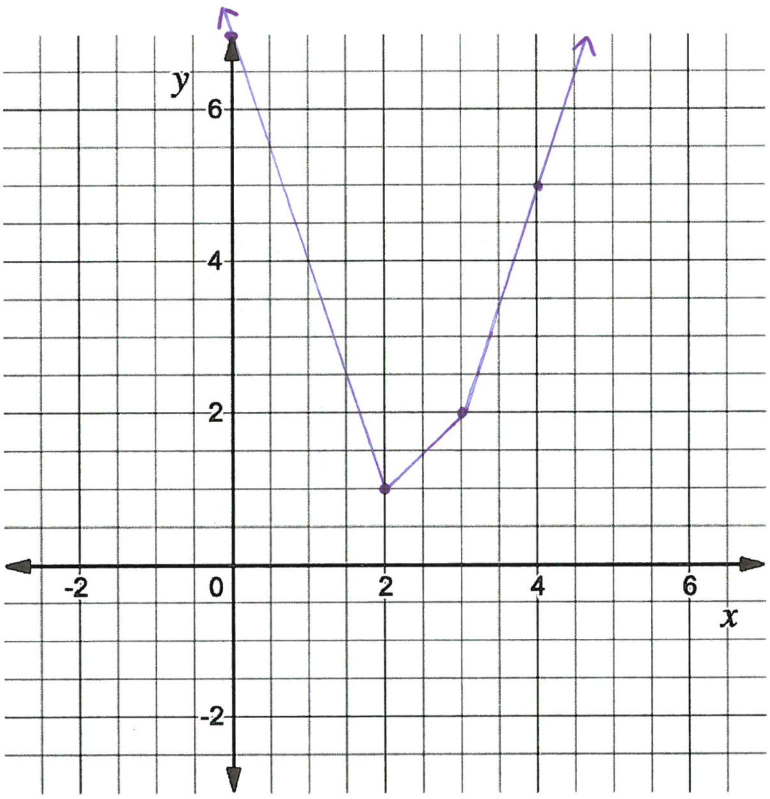
$= 3x-7$

✓ identifies fracture points (for critical regions).

✓ correct pieces ✓ continuous domain

$$\therefore f(g(x)) = \begin{cases} -3x+7, & x \leq 2 \\ x-1, & 2 < x \leq 3 \\ 3x-7, & x > 3 \end{cases}$$

(c) Graph $y = f(g(x))$ on the axes provided below and hence, state its range. (3 marks)



$R_{fog} = \{y : y \in \mathbb{R}, y \geq 1\}$

✓ identifies correct composite range

✓ accuracy at fracture pts. $(2,1)$ and $(3,2)$

✓ correct line segments and continuous.

(d) Hence, or otherwise, solve $f(g(x)) \leq 2$. (1 mark)

Using gradient of $-3x+7$ (GRAPHICALLY),

$\frac{5}{3} \leq x \leq 3$.

✓ states correct inequality for x.

OR

Algebraically,

$-3x+7 = 2$

$-3x = -5$

$x = \frac{5}{3}$

$\therefore \frac{5}{3} \leq x \leq 3$ //.

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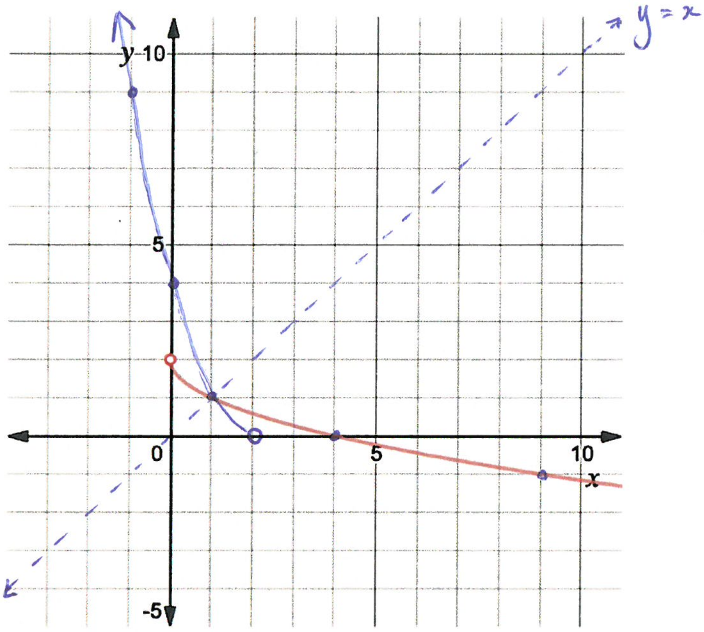
Question 3. **(10 marks)**

Consider the function $f(x) = k - \frac{x}{\sqrt{x}}$. The graph of $y = f(x)$ is shown below.



(a) State the domain of $f(x)$. (1 mark)

$D_f = \{x : x \in \mathbb{R}, x > 0\}$ // ✓ correct domain

(b) State the value of k. (1 mark)

$k = 2$ // ✓ correct k value.

(c) Show, algebraically, that $f(x)$ is indeed a one-to-one function. (2 marks)

If one-to-one, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

$2 - \sqrt{x_1} = 2 - \sqrt{x_2}$ ✓ uses injectivity property

$-\sqrt{x_1} = -\sqrt{x_2}$ ✓ shows one-to-one algebraically.

$x_1 = x_2$ $\therefore$ $f(x)$ is one-to-one //.

(d) Sketch $y = f^{-1}(x)$ on the same axes above and hence, solve $f(x) = f^{-1}(x)$. (3 marks)

✓ shows reflection in line $y = x$

✓ accuracy at $(2,0)$, $(1,1)$, $(0,4)$ and $(-1,9)$

$x = 1$ ✓ Solves for intersection point

(e) State the defining rule for $y = f^{-1}(x)$, including an appropriate restriction on the domain. (3 marks)

Let $y = 2 - \sqrt{x}$

$x \Leftrightarrow y$; $x = 2 - \sqrt{y}$ ✓ interchanges x and y.

$x - 2 = -\sqrt{y}$

$y = (x-2)^2$ ✓ obtains f^{-1}

$\therefore$ $f^{-1}(x) = (x-2)^2$, $D_{f^{-1}} = \{x : x \in \mathbb{R}, x < 2\}$ ✓ states domain restriction.

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Question 4. **(8 marks)**

Consider the rational function, $f(x) = \frac{x^3-2x^2+x+4}{x^2-4}$.

(a) Sketch the rational function $y = f(x)$ on the axes provided below, labelling all critical points. *You do not need to locate the stationary points of the rational function.* (5 marks)

y-intercept : $f(0) = \frac{4}{-4} = -1 \Rightarrow (0,-1)$

x-intercept(s) : $0 = x^3 - 2x^2 + x + 4$

$$\begin{array}{r|rrrr} -1 & 1 & -2 & 1 & 4 \\ & & -1 & 3 & -4 \\ \hline & 1 & -3 & 4 & 0 \end{array}$$

$0 = (x+1)(x^2-3x+4)$

$\therefore\ x = -1$ & no real solns.

$\Rightarrow (-1, 0)$

Vertical asymptote(s): $x = \pm 2$

oblique asymptote : $\dfrac{x(x^2-4) - 2(x^2-4) + 5x - 4}{x^2-4}$

$f(x) = x - 2 + \dfrac{5x-4}{x^2-4}$ → [f(x) crosses asymptote at (0.8, -1.2).]

$\therefore$ oblique at $y = x - 2$

As $x \to \infty$, $f(x) \to \infty$ $[(x-2)^+]$

$x \to -\infty$, $f(x) \to -\infty$ $[(x-2)^-]$

$f(x) = \dfrac{(x+1)(x^2-3x+4)}{(x+2)(x-2)}$

As $x \to 2^+$,

$f(x) = \dfrac{(+)(+)}{(+)(+)}$

$\Rightarrow (+)$

As $x \to 2^-$

$f(x) = \dfrac{(+)(+)}{(+)(-)}$

$\Rightarrow (-)$

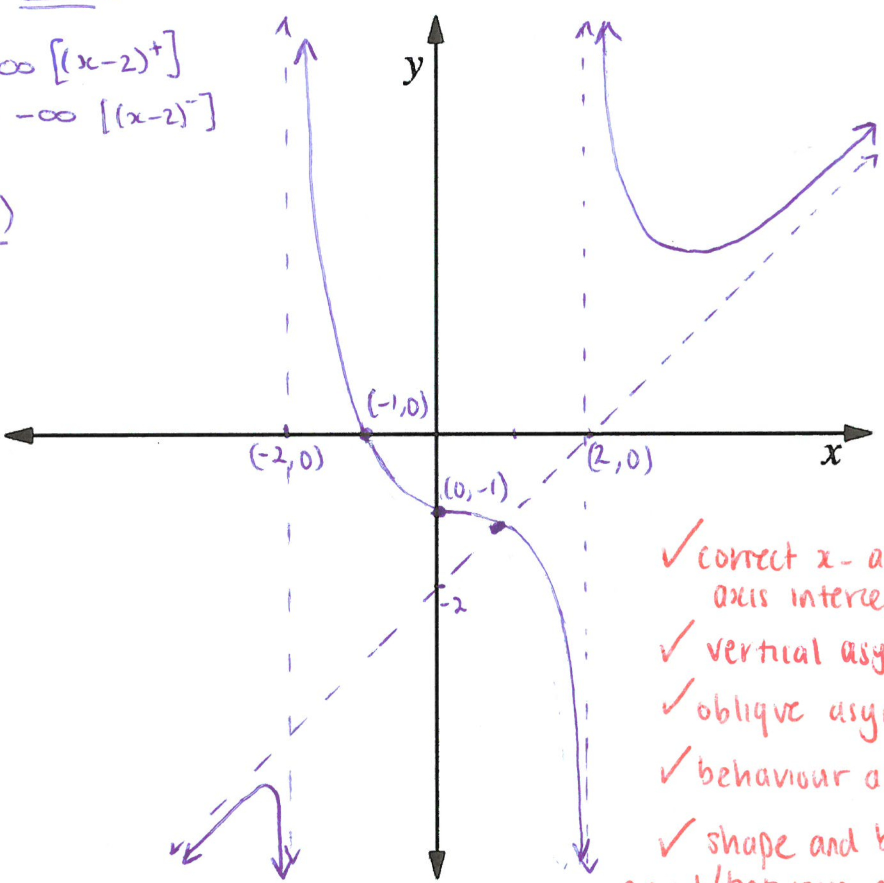
As $x \to -2^+$

$f(x) = \dfrac{(-)(+)}{(+)(-)}$

$\Rightarrow (+)$

As $x \to -2^-$

$f(x) = \dfrac{(-)(+)}{(-)(-)} \Rightarrow (-)$



✓ correct x- and y-axis intercepts

✓ vertical asymptotes

✓ oblique asymptote

✓ behaviour as $x \to \pm\infty$

✓ shape and behaviour around/between asymptotes

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NOTE: *For part (b), two sets of blank axes has been provided, however, there are no marks assigned to a sketch.*

(b) Hence, determine the value(s) of x such that:

(i) $f(x) = |f(x)|$. (2 marks)

$-2 < x \leq -1$ ✓

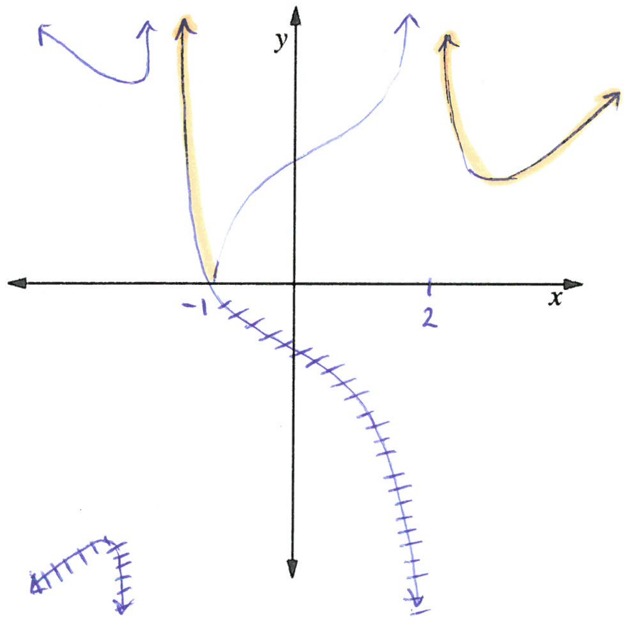
ie. $(-2, -1]$

and

$x > 2$ ✓

ie. $(2, \infty)$

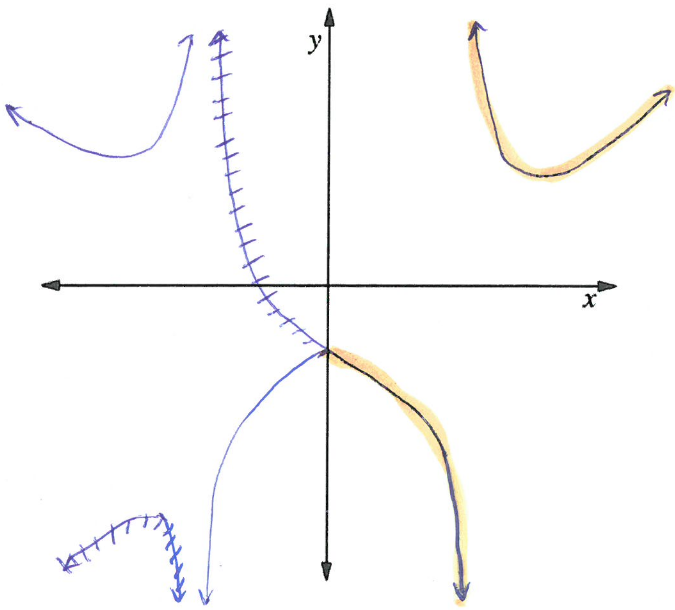
[−1 if incorrect boundary notation]



(ii) $f(x) = f(|x|)$. (1 mark)

$x \geq 0$ ✓

ie $[0, \infty)$



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Question 5. **(7 marks)**

A plane is defined by the vector equation $\Pi_1 : \boldsymbol{r} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} -4 \\ 1 \\ 5 \end{pmatrix}$.

(a) Determine the equation of the plane in the form $\boldsymbol{r} \cdot \boldsymbol{n} = k$. (3 marks)

$$\underset{\sim}{n} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \times \begin{pmatrix} -4 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} -11 \\ -19 \\ -5 \end{pmatrix}$$

✓ uses cross product to find $\underset{\sim}{n}$.

$\cancel{3} \quad \cancel{-4}$
$-2 \times 1 = -10-1$
1×5
$3 \times -4 = -4-15$
$-2 \times 1 = 3-8$
$\cancel{1} \quad \cancel{5}$

$$\therefore \underset{\sim}{r} \cdot \begin{pmatrix} -11 \\ -19 \\ -5 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -11 \\ -19 \\ -5 \end{pmatrix}$$
$$= -22 + 19 - 5$$
$$= -8$$

✓ substitutes into $\underset{\sim}{r} \cdot \underset{\sim}{n} = \underset{\sim}{a} \cdot \underset{\sim}{n}$

$$\therefore \underset{\sim}{r} \cdot \begin{pmatrix} -11 \\ -19 \\ -5 \end{pmatrix} = -8$$

[or $\underset{\sim}{r} \cdot \begin{pmatrix} 11 \\ 19 \\ 5 \end{pmatrix} = 8$ and equivalent.]

✓ states vector equation.

A second plane has a Cartesian equation defined by $\Pi_2 : x + 2y + z = -2$.

Π_1 and Π_2 intersect along a line defined by the equation $ay + bz = d$, where $a, b, d \in \mathbb{Z}$.

(b) Determine a possible set of values for a, b and d. (4 marks)

let $\underset{\sim}{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow -11x - 19y - 5z = -8$

$\Pi_1 : 11x + 19y + 5z = 8$ ① ✓ obtains cartesian equation of Π_1.

$\Pi_2 : x + 2y + z = -2$ $(\times 11)$

$11x + 22y + 11z = -22$ ②

① − ② : $-3y - 6z = 30$ ✓ uses an algebraic method to eliminate x.

or

$3y + 6z = -30$

or

$y + 2z = -10$ ✓ obtains a linear equation $ay + bz = d$.

$\therefore a = 1,\ b = 2$ and $d = -10$

(or equivalent). ✓ states a, b, d.

ALTERNATIVE:

If $\underset{\sim}{r} = \underset{\sim}{a} + \lambda \underset{\sim}{d}$,

✓ 1 mark for finding $\underset{\sim}{d} = \underset{\sim}{n}_1 \times \underset{\sim}{n}_2$

✓ 1 mark for pt. on both planes.

✓ 1 mark for setting $\underset{\sim}{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and equating λ

✓ 1 mark for a, b, d.

End of Calculator Free Section

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Year 12 Mathematics: Specialist
Term 2 2020

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Test 2 *Calculator Assumed*
Functions, Graphs & Vectors in 3D

Student Name: Solutions (FINAL)

Teacher: Alfonsi Moore

Working Time: 25 minutes

Formula Sheet provided
1 A4 page of notes (2 sided)

*Attempt **all** questions.*
*All necessary working and reasoning must be shown for **full marks**.*

Total Marks
$\overline{29}$

Question 6. **(7 marks)**

The vectors in $\mathbb{R}^3$, $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{w}$, have been defined below.

$$\boldsymbol{u} = 3\boldsymbol{i} - 5\boldsymbol{j} + \boldsymbol{k} \qquad \boldsymbol{v} = -\boldsymbol{i} + 10\boldsymbol{j} - 8\boldsymbol{k} \qquad \boldsymbol{w} = 5\boldsymbol{i} - 2\boldsymbol{k}$$

Determine:

(a) $\boldsymbol{u} + \boldsymbol{v} - 2\boldsymbol{w}$ $= -8\underset{\sim}{i} + 5\underset{\sim}{j} - 3\underset{\sim}{k}$ ✓ (1 mark)

(b) $|3\boldsymbol{u} + 2\boldsymbol{v}|$, leaving your answer exact. $= 9\sqrt{3}$ ✓ (1 mark)

(c) $(-2\boldsymbol{w}) \cdot (-\boldsymbol{u})$ $= 26$ ✓ (1 mark)

(d) $\left(\frac{1}{2}\boldsymbol{v}\right) \times \boldsymbol{w}$ (1 mark)

$= -10\underset{\sim}{i} - 21\underset{\sim}{j} - 25\underset{\sim}{k}$ ✓

For the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$,

(e) Determine the angle between the two vectors, rounded to the nearest degree. (1 mark)

143° // ✓

(f) State whether the scalar projection of $\boldsymbol{u}$ onto $\boldsymbol{v}$ would be positive or negative. Justify your answer. (2 marks)

Negative ✓, as angle between the vectors is obtuse ✓

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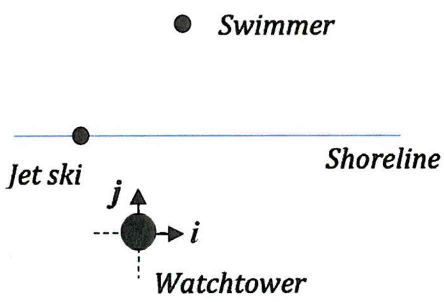
Question 7. **(8 marks)**

From the position of a surf-lifesaving watchtower, a swimmer is spotted caught in a rip with a relative position vector of $\boldsymbol{r_S} = 55\boldsymbol{i} + 400\boldsymbol{j}$ metres.



The rip pulls the swimmer out to sea with a velocity of $\boldsymbol{v_S} = -0.2\boldsymbol{i} + \boldsymbol{j}$ metres per second.

At the instant the swimmer is first spotted, the order is given for a surf-lifesaving jet ski to leave from the shoreline with a relative position of $\boldsymbol{r_J} = -60\boldsymbol{i} + 150\boldsymbol{j}$ metres from the watchtower.

The velocity of the jet ski, taking into account the current, is $\boldsymbol{v_J} = 7\boldsymbol{i} + 16\boldsymbol{j}$ metres per second.

(a) Show that the jet ski will not collide with the swimmer. (3 marks)

$\underset{\sim}{r}_S = \begin{pmatrix} 55 \\ 400 \end{pmatrix} + \lambda \begin{pmatrix} -0.2 \\ 1 \end{pmatrix}$

$\underset{\sim}{r}_J = \begin{pmatrix} -60 \\ 150 \end{pmatrix} + \mu \begin{pmatrix} 7 \\ 16 \end{pmatrix}$ ✓ obtains $\underset{\sim}{r}_S$ and $\underset{\sim}{r}_J$

$\underset{\sim}{r}_S = \underset{\sim}{r}_J \Rightarrow 55 - 0.2\lambda = -60 + 7\mu$ ①

$400 + \lambda = 150 + 16\mu$ ②

✓ uses equations to find scalar coefficients

$\lambda = 8.82$ and $\mu = 16.18$ $\quad \lambda \neq \mu \Rightarrow$ No collision

✓ shows no collision.

OR Assume $\lambda = \mu = t$.

$55 - 0.2t_1 = -60 + 7t_1$

$\Rightarrow t_1 = 15.97$ secs.

$400 + t_2 = 150 + 16t_2$

$\Rightarrow t_2 = 16.67$ secs.

$t_1 \neq t_2$

(b) *Assuming both the swimmer and jet ski remain on the same paths with the same velocities, determine:*

(i) the time taken for the jet ski to get to the point that is closest to the swimmer and state this minimum distance. (3 marks)

$D(t) = |\underset{\sim}{r}_S(t) - \underset{\sim}{r}_J(t)|$

$= \left| \begin{pmatrix} -7.2t + 115 \\ -1.5t + 250 \end{pmatrix} \right|$

$= \sqrt{(115 - 7.2t)^2 + (250 - 1.5t)^2}$

✓ obtains an eqn for distance w.r.t time

$\frac{dD}{dt} = 0 \Rightarrow t = 16.54$ secs.

✓ solves for t

$D(16.54) = 4.51$ m

✓ solves for D.

(ii) the position vectors of the swimmer and jet ski at this closest approach. (2 marks)

$\underset{\sim}{r}_S(16.54) = \begin{pmatrix} 51.69 \\ 416.54 \end{pmatrix}$ m ✓ correct $\underset{\sim}{r}_S$

and

$\underset{\sim}{r}_J(16.54) = \begin{pmatrix} 55.76 \\ 414.59 \end{pmatrix}$ m ✓ correct $\underset{\sim}{r}_J$.

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Question 8. **(14 marks)**

A line, plane and a sphere are defined by the following vector equations.

$$L : \boldsymbol{r} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 5 \\ -1 \end{pmatrix} \qquad \Pi : \boldsymbol{r} \cdot \begin{pmatrix} 1 \\ 4 \\ -6 \end{pmatrix} = 2 \qquad S : \left| \boldsymbol{r} - \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \right| = 4$$

For the following questions, round all final answers to two decimal places.

(a) Explain why the line must intersect the sphere at two points and determine the position vectors of the two points of intersection. (4 marks)

Sphere is centred at $(2,-1,1)$ and line contains pt. $(2,-1,1)$. ✓ suitable explanation

$\Rightarrow$ Line passes through centre $\Rightarrow$ line intersects sphere twice.

$$\left|\begin{pmatrix} -3\lambda \\ 5\lambda \\ -\lambda \end{pmatrix}\right| = 4 \quad \text{✓ substitute } L \to S.$$

$$35\lambda^2 = 16$$

$$\lambda = \pm \frac{4}{\sqrt{35}} \quad (\pm 0.6761)$$

$$\lambda = \pm \frac{4\sqrt{35}}{35} \quad \text{✓ solves for } \lambda$$

$$\therefore \underset{\sim}{r}\left(\frac{4\sqrt{35}}{35}\right) = \begin{pmatrix} -0.03 \\ 2.38 \\ 0.32 \end{pmatrix} //$$

and

$$\underset{\sim}{r}\left(\frac{-4\sqrt{35}}{35}\right) = \begin{pmatrix} 4.03 \\ -4.38 \\ 1.68 \end{pmatrix} //$$

✓ both position vectors correct.

(b) Determine the acute angle between the line and the plane. (2 marks)

$$\cos\theta = \frac{\begin{pmatrix} -3 \\ 5 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 4 \\ -6 \end{pmatrix}}{\sqrt{35}\sqrt{53}}$$

$$\theta = 57.72°$$

✓ finds angle between $\underset{\sim}{n}$ and line

$$\therefore 90 - 57.72$$

$$= 32.28°$$

✓ finds complementary angle

(c) Determine the position vector of the point of intersection between the line and the plane. (3 marks)

$$\begin{pmatrix} 2-3\lambda \\ -1+5\lambda \\ 1-\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 4 \\ -6 \end{pmatrix} = 2 \quad \text{✓ subs } L \to \Pi.$$

$$23\lambda - 8 = 2$$

$$23\lambda = 10$$

$$\lambda = \frac{10}{23} // \quad (0.4348) \quad \text{✓ solves for } \lambda.$$

$$\underset{\sim}{r}\left(\frac{10}{23}\right) = \begin{pmatrix} 0.70 \\ 1.17 \\ 0.57 \end{pmatrix} \quad \text{✓ correct position vector.}$$

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Recall the equations of the plane and sphere below.

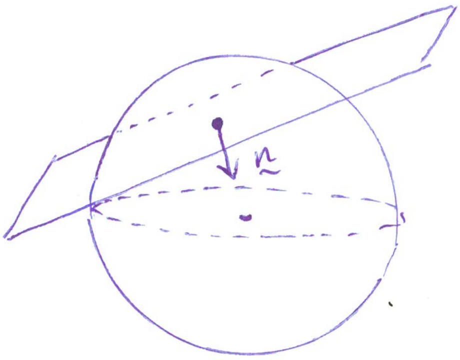
$$\Pi : \boldsymbol{r} \cdot \begin{pmatrix} 1 \\ 4 \\ -6 \end{pmatrix} = 2 \qquad\qquad S : \left| \boldsymbol{r} - \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \right| = 4$$

(d) Determine whether or not the plane intersects the sphere.

If so, determine the radius of the intersection circle formed by the plane and the sphere.

If not, determine the shortest distance from the plane to the surface of the sphere.

(5 marks)



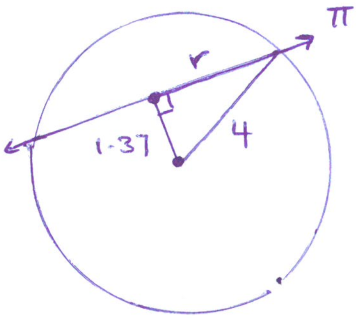
Construct a line through centre of sphere, in the direction of the normal.

$\underset{\sim}{r} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 4 \\ -6 \end{pmatrix}$ ✓obtains line through centre

Intersection between line and plane occurs when

$$\begin{pmatrix} 2+\mu \\ -1+4\mu \\ 1-6\mu \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 4 \\ -6 \end{pmatrix} = 2$$

$2+\mu-4+16\mu-6+36\mu = 2 \Rightarrow \mu = \frac{10}{53}$ (0.1887).



$\underset{\sim}{r}\left(\frac{10}{53}\right) = \begin{pmatrix} 2.19 \\ -0.25 \\ -0.13 \end{pmatrix}$ is the pt on the plane, closest to the centre of the sphere. ✓finds point on plane and line (or μ)

$\left| \begin{pmatrix} 2.19 \\ -0.25 \\ -0.13 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \right| = 1.37$ ✓calculates distance from centre to plane.

$1.37 < 4 \therefore$ point is inside sphere

$\Rightarrow \Pi$ intersects S.

✓ $d < R \therefore$ intersects.

$r = \sqrt{4^2 - 1.37^2}$

$= 3.76$ units ✓calculates r of circle.

[OR : Find any pt. on plane e.g. $\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$ ✓1st mark

$\underset{\sim}{v} \Rightarrow$ vector from centre to point $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ ✓2nd mark

$|\text{proj}_{\underset{\sim}{n}} \underset{\sim}{v}| = 1.37 < 4$ ✓3rd]

End of Calculator Assumed Section